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SHAPES OF ANNULAR LAYERS OF FLUID ON THE SURFACE OF A ROTATING CYLINDER

V. E. Epikhin, P. N. Konon, and V. Ya. Shkadov

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A qualitative and quantitative study is made of the equilibrium forms of plane and axisymmetric fluid layers.

The power-engineering, chemical, and building sectors make use of production processes based on the phenomenon of instability of the free surface of a layer of fluid. For example, the production of thermal insulating wool by the centrifugal-roller method involves the disintegration of a layer formed on the surface of a rapidly rotating cylinder when a mineral melt falls onto the roller [1]. The study [2] presented photographs reflecting the stages of formation of layers of a viscous fluid (such as glycerin or aqueous solutions of glycerin) obtained on an experimental unit which included a rotating cylindrical roller mounted on the horizontal shaft of an electric motor. Some of the liquid which falls onto the roller is thrown off by centrifugal forces. The rest of the liquid is entrained by the rotating surface in the form of an annular layer, with drops separating from the layer about its entire perimeter. When a certain period of time has elapsed after cessation of the supply of fluid, a steady-state regime is established in which the fluid ring, with a smooth surface, rotates as a solid. With an increase in the speed of rotation, the surface of the ring may acquire a wavy shape — as in the photograph shown in Fig. When the speed is increased above a certain critical value, more of the mass of the fluid is thrown from the roll and another stationary fluid ring with a wavy free surface is established.

The studies [3-7] used the small parameter method to theoretically investigate the forms of equilibrium of liquid streams and layers near bifurcation points. Here, we perform a quantitative and qualitative study of nonlinear solutions in relation to values of the characteristic parameters.

1. Formulation of the Problem and Derivation of the Basic Equation. We introduce a cylindrical coordinate system 0, x, y, φ (Fig. 2). The motion of the viscous fluid is described by the Navier–Stokes equations, the continuity equation, and the equation of the free surface:

$$\frac{d\mathbf{u}}{dt} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{u},\ \nabla\mathbf{u} = 0,\ \rho = \mathrm{const}, \tag{1}$$

$$\frac{dh}{dt} = v,\ y = h(x,\ \varphi,\ t). \tag{2}$$

The normal and shear stresses on the external surface of the layer satisfy the conditions in [8]. Due to adhesion, the components of velocity on the roller surface have the following values:

$$u = 0,\ v = 0,\ w = 1,\ y = 1. \tag{3}$$

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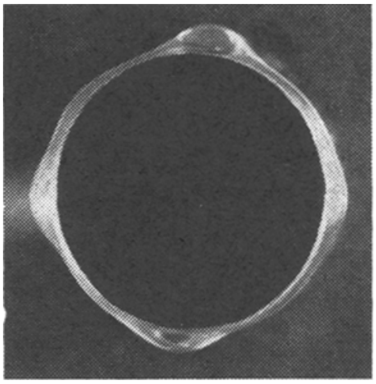

Fig. 1. Photograph of a layer of glycerin on the surface of a cylinder of the radius 0.0123 m, rotating at a speed N = 163 rpm.

We will ignore viscous interaction with the environment and body forces. We change over to the coordinate system τ, ξ, η, θ connected with the rotating cylinder:

$$\tau = t,\ \xi = x,\ \eta = y,\ \theta = \varphi - t.$$

We seek steady-state solutions to problem (1-3) corresponding to a layer which is immobile relative to the surface of the cylinder:

$$u = 0,\ v = 0,\ w = 0,\ h = h(\xi,\ \theta). \tag{4}$$

System (1-2) is satisfied identically if $p(\eta)$ is determined from the equation

$$\frac{\partial p}{\partial \eta} = \eta, \tag{5}$$

the solution of which

$$p(\eta) = p_1 + \frac{1}{2}(\eta^2 - 1),\ p_1 = p(1) = \mathrm{const}. \tag{6}$$

Having inserted Eq. (6) into the boundary condition for the normal stresses, we obtain an equation for determination of the surface of the layer

$$\frac{2}{R_s} = \frac{1}{2}\mathrm{We}(2\mathrm{Eu} + h^2 - 1). \tag{7}$$

Problem (1-3) was examined in [4-6] in the case of plane ($\partial h/\partial x = 0$) and axisymmetric ($\partial h/\partial \varphi = 0$) flows.

2. Plane Layer. In the experiments in [2], the width of the layer was ten times greater than the characteristic value of its thickness. We will assume that the fluid layer is infinite with respect to ξ, so that $h = h(\theta)$. Equation (7) leads us to an equation for $h(\theta)$:

$$2hh'' - 4h'^2 + \mathrm{We}(2\mathrm{Eu} + h^2 - 1)(h'^2 + h^2)^{3/2} - 2h^2 = 0 \tag{8}$$

(the prime denotes differentiation with respect to θ). If we examine a layer with a fixed mass, then Eq. (8) is augmented by the condition

$$\frac{1}{2}\int_0^{2\pi}(h^2 - 1)\,d\theta = M. \tag{9}$$

The constant M is assigned. Let us study the solutions of Eq. (8). For a layer of constant thickness $h = h_0$, Eq. (8) makes it possible to obtain

$$h_0^3 + (2\mathrm{Eu} - 1)h_0 = \frac{2}{\mathrm{We}}. \tag{10}$$

The only root of Eq. (10) $h_0 > 1$ exists for any values of the number We > 0 in the case Eu < 0, while when Eu > 0, it exists under the condition EuWe < 1. Equation (10) leads to the formula

$$2\mathrm{Eu} = 1 + \frac{2}{\mathrm{We}\,h_0} - h_0^2,$$

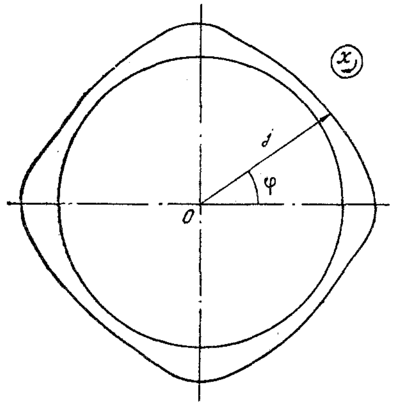

Fig. 2. Free surface of a nonaxisymmetric stationary layer, immobile relative to the surface of the rotating cylinder, in the case We = 9.4, Eu = −0.092, h_1 = 1.10 (calculation).

from which we can determine the pressure on the cylindrical surface $p = p_1$. Linearizing Eq. (8) relative to the equilibrium cylindrical free surface $\eta = h_0$, we obtain the following for perturbations of the surface layer:

$$h'' + \frac{1}{2}[(6h_0\varkappa + 5h_0^3)\,\mathrm{We} - 4]\,h = 0,\ 2\varkappa = 2\mathrm{Eu} - 1. \tag{11}$$

The condition of periodicity of the solution of (11) with respect to θ reduces to satisfaction of the equalities

$$(6h_0\varkappa_n + 5h_0^3)\,\mathrm{We}_n - 4 = 2n^2,\ h_0^3 + 2\varkappa_n h_0 = \frac{2}{\mathrm{We}_n}. \tag{12}$$

Equations (12) make it possible to determine bifurcation points of the parameters We_n and $\varkappa_n$ at which, together with a cylindrical surface of constant thickness, there exist equilibrium cylindrical surfaces whose nornal sections have n-th-order symmetry. We can use (12) to express We_n and $\varkappa_n$ through h_0 and n:

$$\mathrm{We}_n = \frac{n^2 - 1}{h_0^3},\quad \varkappa_n = \frac{(3 - n^2)\,h_0^2}{2\,(n^2 - 1)}, \tag{13}$$

and to express h_0 and $\varkappa_n$ through We and n:

$$\varkappa_n = \frac{3 - n^2}{2\sqrt[3]{\mathrm{We}^2\,(n^2 - 1)}},\quad h_0 = \sqrt[3]{\frac{n^2 - 1}{\mathrm{We}}} > 1 \tag{14}$$

$(n \neq 1,\ 0 > \varkappa_2 > \varkappa_3 = \varkappa_0 > \varkappa_4 > \varkappa_5 > \ldots)$. With allowance for the inequality $h_0 > 1$, Eqs. (14) permit us to obtain the necessary conditions for branching of a cylindrical surface with n-th-order symmetry:

$$(n - 1)^2 - 1 < \mathrm{We} < n^2 - 1,\ \frac{3 - n^2}{\mathrm{We}} < \varkappa < \frac{3 - n^2}{2\,(n^2 - 1)} < 0\ (n = 2, 3, \ldots).$$

Bifurcations were analyzed in [3] without allowance for Eq. (10).

We will examine the surface of a layer of wavy form $\eta = h(\theta)$. Using the substitution of variables $h' = q(h)$, we obtain the first integral of Eq. (8):

$$h'^2 = -h^2\,\frac{Q_1(h,\ B)\,Q_2(h,\ B)}{Q_0^2(h,\ B)},\ Q_1Q_2 \leqslant 0; \tag{15}$$

$$Q_0 = h^4 + 2\,(2\mathrm{Eu} - 1)\,h^2 - B > 0,\ Q_1 = Q_0 + \frac{8h}{\mathrm{We}},\ Q_2 = Q_0 - \frac{8h}{\mathrm{We}}. \tag{16}$$

It is easy to see that if we diffeentiate with respect to θ, we can use integral (15) to obtain the initial equation (8), assuming that $Q_0 > 0$. The below inequalities follow from Eqs. (15) and (16) and the inequality $Q_0 > 0$:

$$Q_2(h,\ B) \leqslant 0 < Q_0(h,\ B) < Q_1(h,\ B).$$

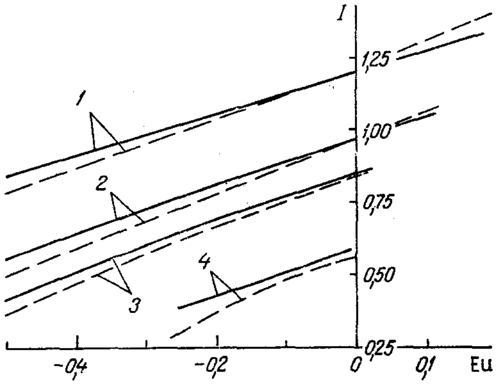

Fig. 3. Dependence of the integral (18) on the number Eu with different values of We: 1) We = 3; 2) 6.25; 3) 9.4; 4) 25. The solid lines correspond to $h_1 = 1.1$, the dashed lines correspond to $h_1 = 1.01$.

The values of h at which $Q_2(h, B) = 0$ and, thus, $h' = 0$, correspond to the minima (valleys) and maximum (crests) of the wavy surface $\eta = h(\theta)$. We obtain the following from integral (15)

$$h' = \pm h \sqrt{-\frac{Q_1(h, B)\,Q_2(h, B)}{Q_0^2(h, B)}}, \quad \theta = \pm \int \frac{Q_0(h, B)\,dh}{h\sqrt{-Q_1(h, B)\,Q_2(h, B)}}. \tag{17}$$

On the wavy surface of the layer $\eta = h(\theta)$, the distance between two successive troughs $h = h_1$ or crests $h = h_2$ determines the wavelength λ, while the distance between a valley and a crest determines the half-wavelength $\lambda/2$ (Fig. 2). Here, $\lambda = 2\pi/n$, where n = 1, 2, ... is the mode of the solution corresponding to the wavy surface. In Eq. (17), the + sign corresponds to the condition $h' \geqq 0$, while the − sign corresponds to $-h' < 0$ on the half-wave in question. The periodicity condition is satisfied if an even number of half-waves is located about the perimeter of the roller, i.e., if the following equation is satisfied

$$\int_{h_1}^{h_2} \frac{Q_0(h, B)\,dh}{h\sqrt{-Q_1(h, B)\,Q_2(h, B)}} = \frac{\pi}{n}. \tag{18}$$

If the root Q_2 is a multiple root, then the condition Q'_{2h} makes it possible to obtain Eq. (10) for a layer of constant thickness.

We will examine a numerical method of constructing periodic solutions of Eq. (8). Let θ_1 be the point of the extremum of the layer thickness, so that $h = h_1$ and $h' = 0$ at $\theta = \theta_1$. Assuming that the parameters We, Eu, and h_1 are known, we express the constant B from the condition $Q_2(h, B) = 0$:

$$B = h_1^4 + 2(2\mathrm{Eu} - 1)\,h_1^2 - \frac{8h_1}{\mathrm{We}}.$$

We fix values of h_1, n, and We and we select an appropriate value of Eu to satisfy Eq. (18). Values of Eu are considered and refined (by the method of division by half, for example) until the integral (18) converges to π/n with the prescribed accuracy. The surface of the layer is determined by numerical integration of Eq. (8) with the same values of θ_1, h_1, We, and n and the value found for Eu. We simultaneously use Eq. (9) to find the mass of the liquid per unit length along the roller axis. In order to keep the fluid mass constant in the periodic solutions of Eq. (8), h_1 and Eu should be chosen so as to simultaneously satisfy Eqs. (18) and (9) with the required accuracy.

The improper integral (18) is changed to the form

$$I = \int_{-1}^{1} \frac{F(h_*)\,dh_*}{\sqrt{1 - h_*^2}}, \quad h_* = \frac{2h - (h_1 + h_2)}{h_2 - h_1}, \tag{19}$$

$$F(h_*) = \frac{Q_0(h_*, B)}{h_*} \sqrt{\frac{(h_* - h_1)(h_* - h_2)}{Q_1(h_*, B)\,Q_2(h_*, B)}}. \tag{20}$$

TABLE 1. Comparison of the Experimental Data in [2] with Calculated Results

Data from [2]					W	From Eqs. (4)			From nonlinear theory		
No. of expt.	$10^2 R_0$, m	ω_0, 1/sec	n	h_s		n	h_s	E	n	h_s	E
1	1,23	33,51	4	1,17	9,40	4	1,168	—0,092	4	1,165	—0,090
2	2,5	18,85	4—6	1,12	24,98	6	1,119	—0,090	6	1,117	—0,088
3	2,5	25,13	7—9	1,12	44,41	7	1,026	—0,005	7	1,027	—0,005
									8	1,098	—0,085
									9	1,135	—0,130
4	2,5	33,51	9—11	1,07	78,95	9	1,004	0,008	9	1,013	—0,008
									10	1,067	—0,058
									11	1,102	—0,100
5	3,5	18,85	7—9	1,06	68,55	9	1,053	—0,040	9	1,052	—0,040
6	3,5	25,13	12—13	1,05	121,9	12	1,055	—0,049	12	1,056	—0,050
									13	1,082	—0,080
7	3,5	33,51	14—16	1,03	216,7	15	1,011	—0,007	15	1,014	—0,010
									16	1,043	—0,040
8	3,5	41,89	17—19	1,03	338,5	19	1,022	—0,018	19	1,021	—0,018

To calculate the integral (19), we use the Hermitian formula:

$$I(F) = \frac{\pi}{K} \sum_{k=1}^{K} F(h_k),\ h_k = \cos\left[\frac{(2k-1)\pi}{2K}\right],$$

where K is the number of theoretical nodes of the function $F(h_*)$, k = 1, 2, ..., K. Let us present the results of calculations performed for annular layers with typical values of the dimensionless parameters. Figure 1 shows a photograph of the surface of a stationary layer in the case We = 37.6.

Before we compare the theoretical results with experimental data, we should note the following. We average the pressure across the layer $1 \leq \eta \leq h(\theta)$ by means of the weight factor e^2:

$$p_* = e^2 p_1 + (1 - e^2)\, p(h),\ 0 \leqslant e^2 \leqslant 1.$$

The basic equation (8) is transformed as follows:

$$2hh'' - 4h'^2 + \mathrm{We}\,[2\mathrm{Eu} - e^2(1-h^2)](h'^2 + h^2)^{3/2} - 2h^2 = 0. \tag{21}$$

Equation (21) leads to (8) if we replace the dimensionless parameters Eu and We by the parameters $E = \mathrm{Eu}/e^2$ and $W = e^2/\mathrm{We}$, respectively. A change in the angular velocity ω_0 by the amount $e\omega_0$ leads to the same change in these parameters. Figures 2 and 3 show results of calculations performed with e = 1/2. Table 1 compares data obtained in [2] with the calculated results. The minimum value of the radius of the layer surface was taken from the photograph.

3. Axisymmetric Layer. In the experiments in [6], a viscous liquid which fell onto the surface of a rotating cylinder with a horizontal axis spread over the cylinder in the form of a layer of constant thickness. Under the influence of random perturbations, the layer acquired a wavelike form and then broke up into rings. The largest of these rings continued to break up until a certain minimum size was reached. The shape of the free surface of the annular layers which formed remained close to axisymmetric. We will assume that the fluid was immobile relative to the surface of the rotating cylinder: u = v = w = 0, $h = h(\xi)$. Equation (7) allows us to obtain the following:

$$2hh'' - 2(h'^2 + 1) + \mathrm{We}\,(2\mathrm{Eu} + h^2 - 1)\, h\, (h'^2 + 1)^{3/2} = 0 \tag{22}$$

(here, the prime denotes differentiation with respect to ξ). Linearizing Eq. (22) relative to the equilibrium cylindrical free surface $\eta = h_0$, we obtain the following for small deviations

$$h'' + \left(\frac{1}{h_0^2} + \mathrm{We}\, h_0\right) h = 0. \tag{23}$$

The periodicity condition of solution (23), with the wavelength $\lambda = 2\pi/\alpha$ (α is the wave number of the perturbed surface), requires satisfaction of the equalities

$$h_0^3 + 2\varkappa h_0 = \frac{2}{\mathrm{We}},\ \frac{1}{h_0^2} + \mathrm{We}\, h_0 = k^2\alpha^2\ (k = 1, 2, \ldots). \tag{24}$$

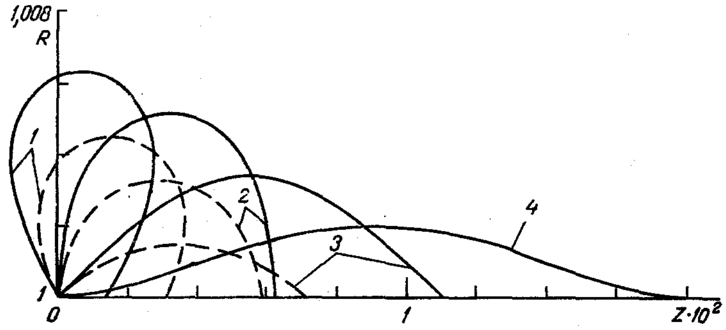

Fig. 4. Calculated forms of the free surface of an axisymmetric layer which is stationary relative to the surface of a rotating layer in the case We = 10^5 and R_0 = 1 with different values of ε_0 and Eu.

System (24) makes it possible to determine the bifurcation points of the parameters We_k and $\varkappa_k$:

$$\mathrm{We}_k = \frac{(k\alpha h_0)^2 - 1}{h_0^3} > 0, \quad \varkappa_k = \frac{[3 - (k\alpha h_0)^2]\, h_0^2}{2\,[(k\alpha h_0)^2 - 1]}. \tag{25}$$

Periodic equilibrium layers were studied in [4] without allowance for condition (10).

Let us study the nonlinear solutions of Eq. (22). We will change over to the internal coordinate system connected with the surface of the layer [9] and transform the problem to the following form:

$$\frac{d\varepsilon}{ds} - \frac{\cos\varepsilon}{R(s)} = -\frac{1}{2}\,\mathrm{We}\,[2\mathrm{Eu} + R^2(s) - 1], \tag{26}$$

$$\frac{dZ}{ds} = \cos\varepsilon, \quad \frac{dR}{ds} = \sin\varepsilon; \tag{27}$$

$$\varepsilon(0) = \varepsilon_0, \; Z(0) = 0, \; R(0) = R_0. \tag{28}$$

Here, s is the length of the arc of a meridional section of the free surface; ε is the tangent angle formed with the symmetry axis at the corresponding point of this section; ε_0 is the angle of contact and is given. It should be noted that system (26), (27) is invariant relative to a substitution of variables

$$s_1 = s, \; Z_1 = L - Z, \; R_1 = R, \; \varepsilon_1 = \pi - \varepsilon, \; \mathrm{We}_1 = -\mathrm{We}. \tag{29}$$

With the replacement of We by −We, Eqs. (26) and (27) describe the free surface of the layer on the inside surface of a rigid cylindrical shell.

The first integral of Eq. (26) has the form

$$R^4 + 2\,(2\mathrm{Eu} - 1)\,R^2 - \frac{8\cos\varepsilon}{\mathrm{We}}\,R = B, \tag{30}$$

where B is the constant of integration. Differentiating both parts of (30) with respect to s and allowing for (27), we find Eq. (26). We solve (30) for cos ε. By virtue of the inequality |cos ε| ≦ 1, we obtain:

$$-1 \leqslant \cos\varepsilon = \frac{\mathrm{We}}{8}\,[R^4 + 2\,(2\mathrm{Eu} - 1)\,R^2 - B]\,\frac{1}{R} \leqslant 1. \tag{31}$$

The first integral of (30) is used to establish characteristic points of the line η = R(s). For example, at extremum points, the equalities ε = 0, π, cos ε= ±1 are satisfied. The integral (30) allows us to obtain

$$R^4 + 2\,(2\mathrm{Eu} - 1)\,R^2 \mp \frac{8}{\mathrm{We}}\,R = B. \tag{32}$$

Stationary axisymmetric layers which are periodic along the generatrix of a cylindrical roller were obtained theoretically in [4]. Figure 4 shows results of numerical integration of system (26), (27) with initial conditions (28) in the case We = 10^5 and R_0 = 1 with different values of ε_0 and Eu. Solid lines 1-4 show the free surface of the layer in the case

$Eu = -0.001$, when the angle ε_0 successively takes the values 112, 80, 40, and 4°. The results agree with a qualitative study made using Eqs. (30) and (32). Dashed lines 1-3 show the free surface in the case $Eu = 0.0008$, when ε_0 takes values of 112, 80, and 40°. An increase in ε_0 above 112° leads to self-intersection of the free surface, similar to [9]. Thus, for example, the calculations showed the following: if $\varepsilon_0 = 128°$, then self-interaction occurs at $Eu = 0.0008$; if $\varepsilon_0 = 136°$, then it occurs at $Eu = 0$; if $\varepsilon_0 = 144°$, then it takes place at $Eu = -0.001$. A reduction in the number Eu to −0.021, −0.041, and −0.061 leads to self-intersection of the surface at just $\varepsilon_0 = 4°$.

NOTATION

Dimensionless parameters of the problem: Reynolds number, Weber number, and Euler number $Re = (\rho R_0^2\omega_0)/\mu$, $We = (\rho R_0^3\omega_0^2)/\sigma$, $Eu = (p_1 - p_a)/(\rho R_0^2\omega_0^2)$; μ, ρ, σ, absolute viscosity, density, and surface tension of the liquid, respectively; p_1, pressure on the surface of the cylinder; p_a, ambient pressure; N, number of roller revolutions per second ($\omega_0 = 2\pi N$); h_s, mean thickness of the layer, determined by the equation $h_s = \sqrt{M/\pi + 1}$; R_s, mean curvature of the layer surface, determined by the expression $2/R_s = [(1 + h_x^2)(h^2 + 2h_\varphi^2 - hh_{\varphi\varphi}) - 2h_x h_\varphi (h_x h_\varphi - hh_{x\varphi}) - hh_{xx}(h^2 + h_\varphi^2)]/(h^2 + h_\varphi^2 + h^2h_x^2)^{3/2}$, where we used the notation: $h_x = \partial h/\partial x$, $h_\varphi = \partial h/\partial\varphi$, $h_{x\varphi} = \partial^2 h/\partial x\partial\varphi$, $h_{xx} = \partial^2 h/\partial x^2$, $h_{\varphi\varphi} = \partial^2 h/\partial\varphi^2$; $\mathbf{u} = \{u, v, w\}$, u, v, w, axial, radial and transverse components of velocity; p, pressure in the layer; L, constant.

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